MTH 605: Topology I
Assignment 3

1 Problems for practice

1.1 Homotopy theory and Fundamental groups

(1) Reading Assignment: Read Lemma 54.2, Theorem 55.8, Theorem 57.1, and Theorem 57.4 from Munkres (2^{nd} Ed.).

(2) Prove the assertions (written in blue) left for verification from the solutions to Quiz 1 and the Midterm.

(3) A space X is *contractible* if the identity map on X is nullhomotopic.

   (a) Show that $\mathbb{R}^n$ is contractible.

   (b) Show that any contractible space is path connected.

(4) Let x_0 and x_1 be points in a path-connected space X. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair f and g of paths from x_0 to x_1, we have $\hat{f} = \hat{g}$.

(5) For $A \subset X$, a continuous map $r : X \to A$ such that $r|_A = i_A$ is called a *retraction* of X into A. If $a \in A$, show that $r_* : \pi_1(X, a) \to \pi_1(A, a)$ is surjective.

(6) Let A be a subspace of $\mathbb{R}^n$, and let $h : (A, a) \to (Y, y)$. Show that if h is extendable to a continuous map of $\mathbb{R}^n$ into Y, then h_* is trivial.

(7) Assuming that there is no retraction $r : D^{n+1} \to S^n$, show that every continuous map $f : D^n \to D^n$ has a fixed point.

(8) Show that if A is a retract of the D^2, then every continuous map $f : A \to A$ has a fixed point.

(9) Compute the fundamental group of the complement of n lines through the origin in $\mathbb{R}^3$.

1.2 CW-complexes and the Seifert-Van Kampen Theorem

(1) Let S_g be the closed orientable surface of genus $g \geq 2$. Consider a separating curve S in S_g that separates S_g into subsurfaces S_{g_1} and S_{g_2} as shown in the figure below, and a nonseparating curve C in S_g (as indicated).

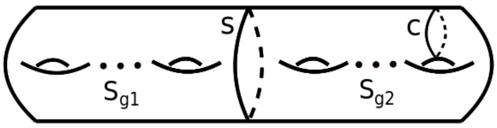

Figure 1: The surface S_g.

(a) Show that S_g does not retract onto the curve S. [Hint: Does S_{g_2} retract onto S?]

(b) Show that S_g retracts onto the curve C.

(2) Let S_g denote the closed orientable surface of genus $g \geq 1$, and for $b \geq 1$ let $S_{g,b}$ be the surface S_g with b boundary components (i.e. S_g with the interiors of b disjoint disks removed).

(a) What is $\pi_1(S_{g,1})$? Using $\pi_1(S_{1,1})$ and induction, derive a presentation for $\pi_1(S_g)$ using the Seifert van Kampen theorem.

(b) Show that there exists a natural epimorphism $\varphi : \pi_1(S_{g,1}) \to \pi_1(S_g)$. What is kernel of this epimorphism?

(3) Consider the quotient space X obtained from the solid cube $I^3 = [0,1]^3$ by identifying each square face to its opposite square face via a right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter right twist of the face about its center point.

(a) Show that X admits a cell complex structure with two 0-cells, four 1-cells, three 2-cells, and one 3-cell.

(b) Using this structure prove that $\pi_1(X)$ is isomorphic to the quaternion group of order 8.

(4) Let X be a topological space. The *mapping torus* M_f of a map $f : X \to X$ is the quotient of $X \times I$ obtained by identifying each point $(x, 0)$ with $(f(x), 1)$.

(a) Assuming that f is basepoint-preserving, compute a presentation for $\pi_1(M_f)$ in terms of the induced map f_* for the following spaces.

(i) $X = S^1 \vee S^1$

(ii) $X = S^1 \times S^1$

(b) Let X be a compact and connected surface and $id : X \to X$ is the identity map. What is $\pi_1(M_{id})$?

(c) Suppose that $f, f' : X \to X$ are homeomorphisms such that $f \simeq f'$, then is $M_f \approx M_{f'}$?

(5) Put a cell complex structure on the following spaces and compute their fundamental groups.

(a) The quotient space of S^2 obtained by identifying the north and the south poles.

(b) The space obtained by taking two copies of the torus $S^1 \times S^1$ and by identifying the circle $S^1 \times \{1\}$ with the circle $S^1 \times \{1\}$ of the other.

(c) For $g \geq 0$ and $b \geq 1$, the closed orientable surface $S_{g,b}$ of genus g with b boundary components (i.e. $S_{g,b}$ is homeomorphic to the surface obtained by the deleting b disjoint open disks from S_g.)

1.3 Covering spaces

(1) Let X be a connected space and $p : \widetilde{X} \to X$ be a covering such that $p^{-1}(x_0)$ has k elements for some $x_0 \in X$. Then show that $p^{-1}(x)$ has k elements for every $x \in X$. (Such an $\widetilde{X}$ is called an *k-fold or k-sheeted covering space* of X.)

(2) Show that $p_n : S^1(\subset \mathbb{C}) \to S^1(\subset \mathbb{C})$ given by $p_n(z) = z^n$ is an n-fold covering space for every positive integer n. Show that these are the only finite-sheeted covers of S^1.

(3) Let $p : \widetilde{X} \to X$ be continuous and surjective. Suppose that U is open set in X that is evenly covered by p. Then show that if U is connected, then the partition of $p^{-1}(U)$ to slices is unique.

(4) Let $p : \widetilde{X} \to X$ is a covering map. Show that if $\widetilde{X}$ is path-connected and X is simply-connected, then p is a homeomorphism.

(5) Let $p : \widetilde{X} \to X$ be a covering space with $p^{-1}(x)$ finite for all $x \in X$. Show that $\widetilde{X}$ is compact Hausdorff iff X is compact Hausdorff.

(6) Show that if a path-connected, locally path-connected covering space X has a finite fundamental group, then every map $f : X \to S^1$ is nullhomotopic.

(7) Consider the map $p \times i : \mathbb{R} \times \mathbb{R}_+ \to S^1 \times \mathbb{R}_+$, where i is the identity map of $\mathbb{R}_+$ and $p : \mathbb{R} \to S^1$ is the universal covering space.

(i) Show that $p \times i : \mathbb{R} \times \mathbb{R}_+ \to S^1 \times \mathbb{R}_+$ is a covering space.

(ii) Sketch the paths $f(t) = (2 - t, 0)$, $g(t) = (1 + t)\cos 2\pi t, (1 + t)\sin 2\pi t)$, and $h(t) = f * g$, and also their liftings under the covering space above.

1.4 Covering actions and Classification of covering spaces

(1) Let X be the figure-8 space and let $\pi_1(X) = \langle a, b \rangle$.

(a) Find all connected 2-sheeted, 3-sheeted, and 4-sheeted covering spaces of the figure 8 space $S^1 \vee S^1$ up to isomorphism.

(b) Find the covering spaces of $S^1 \vee S^1$ that correspond to the subgroups $\langle a \rangle \leq \pi_1(S^1 \vee S^1)$ and $\langle b \rangle \leq \pi_1(S^1 \vee S^1)$.

(c) Describe the covering of X corresponding to the subgroup of $\pi_1(X)$ generated by the set $\{b^n a b^{-n} : n \in \mathbb{Z}\})$. Is this a regular cover?

(d) Describe the covering of X corresponding to the subgroup of $\pi_1(X)$ generated by the set $\{b^n a b^{-n} : n \in \mathbb{N}\} < \pi_1(X)$. Is this a regular cover?

(e) Let $Y = (\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z})$ be the infinite integer grid.

(i) Show that Y is a infinite-sheeted cover of X.

(ii) What is the subgroup of $\pi_1(X)$ that Y corresponds to? Is it finitely generated?

(2) Consider the torus $T^2 = S^1 \times S^1$.

(a) Describe all k-fold covering spaces of T^2.

(b) Show that every cover of T^2 is homeomorphic to T^2, $S^1 \times \mathbb{R}$, or $\mathbb{R}^2$.

(c) Consider the universal covering space $p^2 : \mathbb{R}^2 \to T^2$. Under this covering map, find a lifting of the loop $\alpha(m, n) = (e^{m\pi s}, e^{n\pi s})$ on the torus to $\mathbb{R}^2$.

(3) Consider the Klein bottle $K = \mathbb{R}P^2 \# \mathbb{R}P^2$.

(a) Describe the universal cover of K.

(b) Show that any finite-sheeted cover of K is either homeomorphic to K or the torus T.

(c) Describe a 2-sheeted cover of K that is homeomorphic to T^2 and identity the normal subgroup of $\pi_1(K)$ corresponding to this cover.

(d) Describe an infinite-sheeted cover of K and identity the normal subgroup of $\pi_1(K)$ corresponding to this cover.

(e) Construct non-normal covering spaces of the Klein bottle by a Klein bottle and by a torus.

(4) Consider the covering space $p : \tilde{X} \to X$ of the wedge X of three circles shown in the figure below.

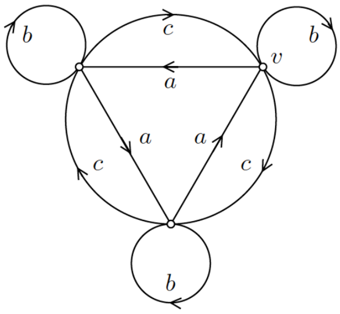

Figure 2: A cover $\tilde{X}$ of the wedge of three circles.

(a) Show that $\tilde{X}$ is a normal cover.

(b) Find a free generating set for the subgroup $p_*(\pi_1(\tilde{X}, v))$ of $\pi(X) = \langle a, b, c \rangle$ that corresponds to this cover.

(5) Let $S_{g,b}$ be the closed oreintable surface of genus g with b boundary components.

(a) Show that for any $k \geq 2$, there exists a k-sheeted regular cover of $p_k : S_{g(k-1)+1} \to S_g$, and hence a k-sheeted regular cover of $S_{g,1} \subset S_g$ given by $p'_k : S_{g(k-1)+1,k} \to S_{g,1}$.

(b) Give presentations for $(p_k)_*(\pi_1(S_{g(k-1)+1})$ and $(p'_k)_*(\pi_1(S_{g(k-1)+1,k}))$.

(c) Prove or disprove the following: There exists an epimorphism $\varphi' : \pi_1(S_{g(k-1)+1,k}) \to \pi_1(S_{g(k-1)+1})$ so that the following diagram commutes.

$$\begin{array}{ccc} \pi_1(S_{g(k-1)+1,k}) & \xrightarrow{\varphi'} & \pi_1(S_{g(k-1)+1}) \\ \big\downarrow{\scriptstyle (p'_k)_*} & & \big\downarrow{\scriptstyle (p_k)_*} \\ \pi_1(S_{g,1}) & \xrightarrow{\quad\varphi\quad} & \pi_1(S_g) \end{array}$$

(6) Describe all the connected covering spaces of $\mathbb{R}P^2 \vee \mathbb{R}P^2$.

2 Problems for submission

- **Homework 3:** Solve problems 1.1 (2), 1.2 (1), and 1.3 (3) from the practice problems above. **(Due 28/3/24)**
- **Homework 4:** Solve problems 1.4 (1) and 1.4 (3) from the practice problems above. **(Due 15/4/24)**